Finite Crystals and Paths

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Dedicated to Professor Tetsuji Miwa on his fiftieth birthday

Abstract. We consider a category of finite crystals of a quantum affine algebra whose objects are not necessarily perfect, and set of paths, semi-infinite tensor product of an object of this category with a certain boundary condition. It is shown that the set of paths is isomorphic to a direct sum of infinitely many, in general, crystals of integrable highest weight modules. We present examples from $C_n^{(1)}$ and $A_{n-1}^{(1)}$, in which the direct sum becomes a tensor product as suggested from the Bethe Ansatz.

1. Introduction

The main object of this note is to define a set of paths from a *finite* crystal B, which is not necessarily perfect, and investigate its crystal structure. The set of paths $\mathcal{P}(\boldsymbol{p}, B)$ is, roughly speaking, a subset of the semi-infinite tensor product $\cdots\otimes B\otimes\cdots\otimes B\otimes B$ with a certain boundary condition related to $\boldsymbol{p}$. If B is perfect, it is known [**KMN1**] that as crystals, $\mathcal{P}(\boldsymbol{p}, B)$ is isomorphic to the crystal base $B(\lambda)$ of an integrable highest weight module with highest weight λ of the quantum affine algebra $U_q(\mathfrak{g})$. While trying to generalize this notion, we had two examples in mind: (a) $\mathfrak{g} = C_n^{(1)}, B = B^{1,l}\ (l : \text{odd})$; (b) $\mathfrak{g} = A_{n-1}^{(1)}, B = B^{1,l}\otimes B^{1,m}\ (l \geq m)$. For this parametrization of finite crystals, we refer to [**HKOTY**]. $B^{1,l}$ stands for the crystal base of an irreducible finite-dimensional $U_q'(\mathfrak{g})$-module. In case (a) (resp. (b)) this finite-dimensional module is isomorphic to $V_{l\overline{\Lambda}_1}\oplus V_{(l-2)\overline{\Lambda}_1}\oplus\cdots\oplus V_{\overline{\Lambda}_1}$ (resp. $V_{l\overline{\Lambda}_1}$) as $U_q(\overline{\mathfrak{g}})$-module, where V_λ is the irreducible finite-dimensional module with highest weight λ. In both cases B is not perfect except when $l = m$ in (b). For precise treatment see section 4.1 for (a) and 4.2 for (b).

Let us consider case (a) first. When $l = 1$ it has already been known [**DJKMO**] that the formal character of $\mathcal{P}(\boldsymbol{p}, B^{1,1})$ for suitable $\boldsymbol{p}$ agrees with that of the irreducible highest weight $A_{2n-1}^{(1)}$-module with fundamental highest weight Λ_i regarded as $C_n^{(1)}$-module via the natural embedding $C_n^{(1)} \hookrightarrow A_{2n-1}^{(1)}$. On the other hand, the Bethe Ansatz suggests [**Ku**] that $\mathcal{P}(\boldsymbol{p}, B^{1,l})$ is equal to $B(\lambda)\otimes\mathcal{P}(\boldsymbol{p}^\dagger, B^{1,1})$ for suitable $\boldsymbol{p}, \boldsymbol{p}^\dagger$ and a level $\frac{l-1}{2}$ dominant integral weight λ at the level of the Virasoro central charge.

Let us turn to case (b). In [**HKMW**] the $U_q'(\widehat{sl}_2)$-invariant integrable vertex model with alternating spins is considered. To translate the physical states and operators of this model into the language of representation theory of the quantum affine algebra $U_q(\widehat{sl}_2)$, they considered a set of paths with alternating spins and showed that it is isomorphic to the tensor product of crystals with highest weights. Another appearance of example (b) can be found in [**HKKOTY**]. They considered the inductive limit of $(B^{1,l})^{\otimes L_1} \otimes (B^{1,m})^{\otimes L_2}$ when $L_1, L_2 \to \infty, L_1 \equiv r_1, L_1 + L_2 \equiv r_2 \pmod{n}$, and showed that there is a weight preserving bijection between the limit and $B((l-m)\Lambda_{r_1}) \otimes B(m\Lambda_{r_2})$. Since there is a natural isomorphism $B^{1,l} \otimes B^{1,m} \simeq B^{1,m} \otimes B^{1,l}$, the above result claims that $\mathcal{P}(\boldsymbol{p}, B^{1,l} \otimes B^{1,m})$ for suitable $\boldsymbol{p}$ is bijective to $B((l-m)\Lambda_{r_1}) \otimes B(m\Lambda_{r_2})$ with weight preserved. These results are consistent with the earlier Bethe ansatz calculations on "mixed spin" models [**AM, DMN**].

If we forget about the degree of the null root δ from weight, this phenomenon is explained using the theory of crystals with core [**KK**]. (See also [**HKMW**] section 3.2.) Let $\{B_k\}_{k \geq 1}$ be a coherent family of perfect crystals and B'_m be a perfect crystal of level m. Fix l such that $l \geq m$ and take dominant integral weights λ and μ of level $l-m$ and m. Then there exists an isomorphism of crystals:

$$\begin{aligned} B(\lambda) \otimes B(\mu) &\simeq B(\sigma\lambda) \otimes B_{l-m} \otimes B(\sigma'\mu) \otimes B'_m \\ &\simeq B(\sigma\lambda) \otimes B(\sigma\sigma'\mu) \otimes (B_l \otimes B'_m), \end{aligned}$$

where σ and σ' are automorphisms on the weight lattice P related to $\{B_k\}_{k\geq 1}$ and B'_m. Iterating this isomorphism infinitely many times, we can expect

$$\mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B_l \otimes B'_m) \simeq B(\lambda) \otimes B(\mu)$$

as $P/\mathbf{Z}\delta$-weighted crystals with suitable $\boldsymbol{p}^{(\lambda,\mu)}$.

In both cases (a),(b) we have illustrated above, what we expect is an isomorphism of P-weighted crystals of the following type:

$$\mathcal{P}(\boldsymbol{p}, B) \simeq B(\lambda) \otimes \mathcal{P}(\boldsymbol{p}^\dagger, B^\dagger) \tag{1.1}$$

and we shall prove it in this paper. First we examine the crystal structure of $\mathcal{P}(\boldsymbol{p}, B)$ and show it is isomorphic to a direct sum of $B(\lambda)$'s. Therefore, the structure of $\mathcal{P}(\boldsymbol{p}, B)$ is completely determined by the set of highest weight elements. In the LHS of (1.1), such set $\mathcal{P}(\boldsymbol{p}, B)_0$ is easy to describe, and in the RHS, this set turns out to be the set of restricted paths $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^\dagger, B^\dagger)$, which is familiar to the people in solvable lattice models. Thus establishing a weight preserving bijection between $\mathcal{P}(\boldsymbol{p}, B)_0$ and $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^\dagger, B^\dagger)$ directly, we can show (1.1).

2. Crystals

2.1. Notation. Let $\mathfrak{g}$ be an affine Lie algebra. We denote by I the index set of its Dynkin diagram. Note that 0 is included in I. Let α_i, h_i, Λ_i $(i \in I)$ be the simple roots, simple coroots, fundamental weights for $\mathfrak{g}$. Let $\delta = \sum_{i \in I} a_i \alpha_i$ denote the standard null root, and $c = \sum_{i \in I} a_i^\vee h_i$ the canonical central element, where $a_i, a_i^\vee$ are positive integers as in [**Kac**]. We assume $a_0 = 1$. Let $P = \bigoplus_{i \in I} \mathbf{Z}\Lambda_i \oplus \mathbf{Z}\delta$ be the weight lattice, and set $P^+ = \sum_{i\in I} \mathbf{Z}_{\geq 0}\Lambda_i \oplus \mathbf{Z}\delta$.

Let $U_q(\mathfrak{g})$ be the quantum affine algebra associated to $\mathfrak{g}$. For the definition of $U_q(\mathfrak{g})$ and its Hopf algebra structure, see e.g. section 2.1 of [**KMN1**]. For $J \subset I$ we denote by $U_q(\mathfrak{g}_J)$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, t_i $(i \in J)$. In particular, $U_q(\mathfrak{g}_{I\setminus\{0\}})$ is identified with the quantized enveloping algebra for the simple Lie algebra whose Dynkin diagram is obtained by deleting the 0 vertex from

that of $\mathfrak{g}$. We also consider the quantum affine algebra without derivation $U'_q(\mathfrak{g})$. As its weight lattice, the classical weight lattice $P_{cl} = P/\mathbf{Z}\delta$ is needed. We canonically identify P_{cl} with $\bigoplus_{i\in I}\mathbf{Z}\Lambda_i \subset P$. For the precise treatment, see section 3.1 of [**KMN1**]. We further define the following subsets of P_{cl}: $P^0_{cl} = \{\lambda \in P_{cl} \mid \langle\lambda, c\rangle = 0\}$, $P^+_{cl} = \{\lambda \in P_{cl} \mid \langle\lambda, h_i\rangle \geq 0$ for any $i\}$, $(P^+_{cl})_l = \{\lambda \in P^+_{cl} \mid \langle\lambda, c\rangle = l\}$. For $\lambda, \mu \in P_{cl}$, we write $\lambda \geq \mu$ to mean $\lambda - \mu \in P^+_{cl}$.

2.2. Crystals and crystal bases. We summarize necessary facts in crystal theory. Our basic references are [**K1**], [**KMN1**] and [**AK**].

A crystal B is a set B with the maps

$$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}$$

satisfying the following properties:

$\tilde{e}_i 0 = \tilde{f}_i 0 = 0$,
for any b and i, there exists $n > 0$ such that $\tilde{e}_i^n b = \tilde{f}_i^n b = 0$,
for $b, b' \in B$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

If we want to emphasize I, B is called an I-crystal. A crystal can be regarded as a colored oriented graph by defining

$$b \xrightarrow{i} b' \quad \Longleftrightarrow \quad \tilde{f}_i b = b'.$$

For an element b of B we set

$$\varepsilon_i(b) = \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{f}_i^n b \neq 0\}.$$

We also define a P-weighted crystal. It is a crystal with the weight decomposition $B = \sqcup_{\lambda\in P} B_\lambda$ such that

$$\tilde{e}_i B_\lambda \subset B_{\lambda+\alpha_i} \sqcup \{0\}, \quad \tilde{f}_i B_\lambda \subset B_{\lambda-\alpha_i} \sqcup \{0\}, \tag{2.1}$$

$$\langle h_i, wt\, b\rangle = \varphi_i(b) - \varepsilon_i(b). \tag{2.2}$$

Set

$$\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i, \quad \varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i.$$

Then (2.2) is equivalent to $\varphi(b) - \varepsilon(b) = wt\, b$. P_{cl}-weighted crystal is defined similarly.

For two weighted crystals B_1 and B_2, the tensor product $B_1 \otimes B_2$ is defined.

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{2.3}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{2.4}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0. ε_i, φ_i and wt are given by

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)), \tag{2.5}$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)), \tag{2.6}$$

$$wt\,(b_1 \otimes b_2) = wt\, b_1 + wt\, b_2. \tag{2.7}$$

Definition 2.1 ([**AK**]). *We say a P (or P_{cl})-weighted crystal is regular, if for any $i,j \in I$ ($i \neq j$), B regarded as $\{i,j\}$-crystal is a disjoint union of crystals of integrable highest weight modules over $U_q(\mathfrak{g}_{\{i,j\}})$.*

Crystal is a notion obtained by abstracting the properties of crystal bases [**K1**]. Let $V(\lambda)$ be the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P^+$ and highest weight vector u_λ. It is shown in [**K1**] that $V(\lambda)$ has a crystal base $(L(\lambda), B(\lambda))$. We regard u_λ as an element of $B(\lambda)$ as well. $B(\lambda)$ is a regular P-weighted crystal. A finite-dimensional integrable $U'_q(\mathfrak{g})$-module V does not necessarily have a crystal base. If V has a crystal base (L,B), then B is a regular P^0_{cl}-weighted crystal with finitely many elements.

Let W be the affine Weyl group associated to $\mathfrak{g}$, and s_i be the simple reflection corresponding to α_i. W acts on any regular crystal B [**K2**]. The action is given by

$$S_{s_i} b = \begin{cases} \tilde{f}_i^{\langle h_i, \mathrm{wt}\, b\rangle} b & \text{if } \langle h_i, \mathrm{wt}\, b\rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, \mathrm{wt}\, b\rangle} b & \text{if } \langle h_i, \mathrm{wt}\, b\rangle \leq 0. \end{cases}$$

An element b of B is called *i-extremal* if $\tilde{e}_i b = 0$ or $\tilde{f}_i b = 0$. b is called *extremal* if $S_w b$ is i-extremal for any $w \in W$ and $i \in I$.

Definition 2.2 ([**AK**] Definition 1.7). *Let B be a regular P^0_{cl}-weighted crystal with finitely many elements. We say B is simple if it satisfies*

(1) *There exists $\lambda \in P^0_{cl}$ such that the weights of B are in the convex hull of $W\lambda$.*
(2) $\sharp B_\lambda = 1$.
(3) *The weight of any extremal element is in $W\lambda$.*

Remark 2.3. *Let B be a regular P^0_{cl}-weighted crystal with finitely many elements. We have the following criterion for simplicity. Let $B(\lambda)$ denote the crystal base of the irreducible highest weight $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module with highest weight λ. If B decomposes into $B \simeq \bigoplus_{j=0}^m B(\lambda_j)$ as $U_q(\mathfrak{g}_{I\setminus\{0\}})$-crystal and λ_j satisfies*

(1) $\lambda_j \in \lambda_0 + \sum_{i\neq 0} \mathbf{Z}_{\leq 0}\alpha_i$ *and* $\lambda_j \neq \lambda_0$ *for any* $j \neq 0$,
(2) *The highest weight element of $B(\lambda_j)$ is not 0-extremal for any $j \neq 0$,*

then B is simple.

Proposition 2.4 ([**AK**] Lemma 1.9 & 1.10). *Simple crystals have the following properties.*

(1) *A simple crystal is connected.*
(2) *The tensor product of simple crystals is also simple.*

2.3. Category $\mathcal{C}^{fin}$. Let B be a regular P^0_{cl}-weighted crystal with finitely many elements. For B we introduce the *level* of B by

$$\mathit{lev}\, B = \min\{\langle c, \varepsilon(b)\rangle \mid b \in B\} \in \mathbf{Z}_{\geq 0}.$$

Note that $\langle c, \varepsilon(b)\rangle = \langle c, \varphi(b)\rangle$ for any $b \in B$. We also set $B_{\min} = \{b \in B \mid \langle c, \varepsilon(b)\rangle = \mathit{lev}\, B\}$ and call an element of $B_{\min}$ *minimal*.

Definition 2.5. *We denote by $\mathcal{C}^{fin}(\mathfrak{g})$ (or simply $\mathcal{C}^{fin}$) the category of crystal B satisfying the following conditions:*

(1) *B is a crystal base of a finite-dimensional $U'_q(\mathfrak{g})$-module.*
(2) *B is simple.*

(3) *For any* $\lambda \in P_{cl}^{+}$ *such that* $\langle c, \lambda\rangle \geq \mathrm{lev}\, B$, *there exists* $b \in B$ *satisfying* $\varepsilon(b) \leq \lambda$. *It is also true for* φ.

We call an object of $\mathcal{C}^{fin}(\mathfrak{g})$ *finite crystal*.

REMARK 2.6. (i) *Condition (1) implies* B *is a regular* P_{cl}^{0}*-weighted crystal with finitely many elements.*

(ii) *Set* $l = \mathrm{lev}\, B$. *Condition (3) implies that the maps* ε *and* φ *from* $B_{\min}$ *to* $(P_{cl}^{+})_l$ *are surjective. (cf. (4.6.5) in* [**KMN1**]*)*

(iii) *Practically, one has to check condition (3) only for* $\lambda \in P_{cl}^{+}$ *such that there is no* $i \in I$ *satisfying* $\lambda - \Lambda_i \geq 0$ *and* $\langle c, \lambda - \Lambda_i\rangle \geq \mathrm{lev}\, B$. *In particular, if* $a_i^{\vee} = 1$ *for any* $i \in I$ *(*$\mathfrak{g} = A_n^{(1)}, C_n^{(1)}$*), the surjectivity of* ε *and* φ *assures (3).*

(iv) *The authors do not know a crystal satisfying (1) and (2), but not satisfying (3).*

Let B_1 and B_2 be two finite crystals. Definition 2.5 (1) and the existence of the universal R-matrix assures that we have a natural isomorphism of crystals.

$$B_1 \otimes B_2 \simeq B_2 \otimes B_1. \tag{2.8}$$

The following lemma is immediate.

LEMMA 2.7. *Let* B_1, B_2 *be finite crystals.*

(1) $\mathrm{lev}\,(B_1 \otimes B_2) = \max(\mathrm{lev}\, B_1, \mathrm{lev}\, B_2)$.

(2) *If* $\mathrm{lev}\, B_1 \geq \mathrm{lev}\, B_2$, *then* $(B_1 \otimes B_2)_{\min} = \{b_1 \otimes b_2 \mid b_1 \in (B_1)_{\min},\ \varphi_i(b_1) \geq \varepsilon_i(b_2)$ *for any* $i\}$.

(3) *If* $\mathrm{lev}\, B_1 \leq \mathrm{lev}\, B_2$, *then* $(B_1 \otimes B_2)_{\min} = \{b_1 \otimes b_2 \mid b_2 \in (B_2)_{\min},\ \varphi_i(b_1) \leq \varepsilon_i(b_2)$ *for any* $i\}$.

$\mathcal{C}^{fin}(\mathfrak{g})$ forms a tensor category.

PROPOSITION 2.8. *If* B_1 *and* B_2 *are objects of* $\mathcal{C}^{fin}(\mathfrak{g})$, *then* $B_1 \otimes B_2$ *is also an object of* $\mathcal{C}^{fin}(\mathfrak{g})$.

Proof. We need to check the conditions in Definition 2.5 for $B_1 \otimes B_2$. (1) is obvious and (2) follows from Proposition 2.4 (2).

Let us prove condition (3) for ε. Set $l_1 = \mathrm{lev}\, B_1, l_2 = \mathrm{lev}\, B_2$. Using (2.8) if necessary, we can assume $l_1 \geq l_2$. Thus we have $\mathrm{lev}\, B_1 \otimes B_2 = l_1$. For any $\lambda \in P_{cl}^{+}$ such that $\langle c, \lambda\rangle \geq l_1$, one can take $b_1 \in B_1$ satisfying $\varepsilon(b_1) \leq \lambda$. Since $\langle c, \varphi(b_1)\rangle \geq l_1 \geq l_2$, one can take $b_2 \in B_2$ satisfying $\varepsilon(b_2) \leq \varphi(b_1)$. In view of (2.5) one has $\varepsilon(b_1 \otimes b_2) = \varepsilon(b_1) \leq \lambda$.

For the proof of φ, repeat a similar exercise for $B_2 \otimes B_1 (\simeq B_1 \otimes B_2)$ using (2.6). ∎

2.4. Category $\mathcal{C}^h$. If an element b of a crystal B satisfies $\tilde{e}_i b = 0$ for any i, we call it a *highest weight* element.

DEFINITION 2.9. *We denote by* $\mathcal{C}^h(I, P)$ *(or simply* $\mathcal{C}^h$*) the category of regular* P*-weighted crystal* B *satisfying the following condition:*

For any $b \in B$, *there exist* $l \geq 0, i_1, \cdots, i_l \in I$ *such that* $b' = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} b \in B$ *is a highest weight element.*

Clearly, $\mathcal{C}^h(I, P)$ forms a tensor category.

PROPOSITION 2.10 ([**KMN1**] Proposition 2.4.4). *An object of $\mathcal{C}^h(I,P)$ is isomorphic to a direct sum (disjoint union) of crystals $B(\lambda)$ ($\lambda \in P^+$) of integrable highest weight $U_q(\mathfrak{g})$-modules.*

Let O be an object of $\mathcal{C}^h(I,P)$. By O_0 we mean the set of highest weight elements in O. Suppose that $O_0 = \{b_j \mid j \in J\}$ and $wt\, b_j = \lambda_j \in P^+$, then from the above proposition we have an isomorphism

$$O \simeq \bigoplus_{j\in J} B(\lambda_j) \quad \text{as } P\text{-weighted crystals.}$$

J can be an infinite set.

The following lemma is standard.

LEMMA 2.11. *Let B_1, B_2 be weighted crystals. Then $b_1 \otimes b_2 \in B_1 \otimes B_2$ is a highest weight element, if and only if b_1 is a highest weight element and $\tilde{e}_i^{\langle h_i, wt\, b_1\rangle+1} b_2 = 0$ for any i.*

Let O be an object of $\mathcal{C}^h(I,P)$. From this lemma we have the following bijection.

$$\begin{array}{ccc} (B(\lambda)\otimes O)_0 & \longrightarrow & O^{\leq\lambda} := \{b \in O \mid \tilde{e}_i^{\langle h_i,\lambda\rangle+1} b = 0 \text{ for any } i\} \\ u_\lambda \otimes b & \mapsto & b. \end{array}$$

Note that $O^{\leq 0} = O_0$.

3. Paths

In this section we construct a set of paths from a finite crystal and consider its structure.

3.1. Energy function. Let us recall the energy function used in [**NY**] to identify the Kostka-Foulkes polynomial with a generating function over classically restricted paths.

Let B_1 and B_2 be two finite crystals. Suppose $b_1 \otimes b_2 \in B_1 \otimes B_2$ is mapped to $\tilde{b}_2 \otimes \tilde{b}_1 \in B_2 \otimes B_1$ under the isomorphism (2.8). A $\mathbf{Z}$-valued function H on $B_1 \otimes B_2$ is called an *energy function* if for any i and $b_1 \otimes b_2 \in B_1 \otimes B_2$ such that $\tilde{e}_i(b_1 \otimes b_2) \neq 0$, it satisfies

$$\begin{aligned} H(\tilde{e}_i(b_1\otimes b_2)) &= H(b_1\otimes b_2)+1 && \text{if } i=0, \varphi_0(b_1)\geq\varepsilon_0(b_2), \\ &&& \quad \varphi_0(\tilde{b}_2)\geq\varepsilon_0(\tilde{b}_1), \\ &= H(b_1\otimes b_2)-1 && \text{if } i=0, \varphi_0(b_1)<\varepsilon_0(b_2), \\ &&& \quad \varphi_0(\tilde{b}_2)<\varepsilon_0(\tilde{b}_1), \\ &= H(b_1\otimes b_2) && \text{otherwise.} \end{aligned} \tag{3.1}$$

When we want to emphasize $B_1 \otimes B_2$, we write $H_{B_1B_2}$ for H. The existence of such function can be shown in a similar manner to section 4 of [**KMN1**] based on the existence of *combinatorial R-matrix.* The energy function is unique up to additive constant, since $B_1 \otimes B_2$ is connected. By definition, $H_{B_1B_2}(b_1\otimes b_2) = H_{B_2B_1}(\tilde{b}_2\otimes\tilde{b}_1)$.

If the tensor product $B_1 \otimes B_2$ is homogeneous, i.e., $B_1 = B_2$, we have $\tilde{b}_2 = b_1, \tilde{b}_1 = b_2$. Thus (3.1) is rewritten as

$$\begin{aligned} H(\tilde{e}_i(b_1 \otimes b_2)) &= H(b_1 \otimes b_2) + 1 \quad \text{if } i = 0, \varphi_0(b_1) \geq \varepsilon_0(b_2), \\ &= H(b_1 \otimes b_2) - 1 \quad \text{if } i = 0, \varphi_0(b_1) < \varepsilon_0(b_2), \\ &= H(b_1 \otimes b_2) \qquad\quad \text{if } i \neq 0. \end{aligned} \tag{3.2}$$

The following proposition, which is shown by case-by-case checking, reduces the energy function of a tensor product to that of each component.

Proposition 3.1. *Set $B = B_1 \otimes B_2$, then*

$$\begin{aligned} H_{BB}((b_1 \otimes b_2) \otimes (b'_1 \otimes b'_2)) &= H_{B_1B_2}(b_1 \otimes b_2) + H_{B_1B_1}(\tilde{b}_1 \otimes b'_1) \\ &\quad + H_{B_2B_2}(b_2 \otimes \tilde{b}'_2) + H_{B_1B_2}(b'_1 \otimes b'_2). \end{aligned}$$

Here $\tilde{b}_1, \tilde{b}'_2$ are defined as

$$\begin{array}{rcl} B_1 \otimes B_2 & \simeq & B_2 \otimes B_1 \\ b_1 \otimes b_2 & \mapsto & \tilde{b}_2 \otimes \tilde{b}_1 \\ b'_1 \otimes b'_2 & \mapsto & \tilde{b}'_2 \otimes \tilde{b}'_1. \end{array}$$

Remark 3.2. *Decomposition of the energy function is not unique. For instance, the following also gives such decomposition.*

$$\begin{aligned} H_{BB}((b_1 \otimes b_2) \otimes (b'_1 \otimes b'_2)) &= H_{B_2B_1}(b_2 \otimes b'_1) + H_{B_1B_1}(b_1 \otimes \check{b}'_1) \\ &\quad + H_{B_2B_2}(\check{b}_2 \otimes b'_2) + H_{B_1B_2}(\check{b}'_1 \otimes \check{b}_2), \end{aligned}$$

where

$$\begin{array}{rcl} B_2 \otimes B_1 & \simeq & B_1 \otimes B_2 \\ b_2 \otimes b'_1 & \mapsto & \check{b}'_1 \otimes \check{b}_2. \end{array}$$

3.2. Set of paths $\mathcal{P}(\boldsymbol{p}, B)$. We shall define a set of paths from any finite crystal in $\mathcal{C}^{fin}$ imitating the construction in section 4 of [**KMN1**] from a perfect crystal.

Definition 3.3. *An element $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_j \otimes \cdots \otimes \boldsymbol{b}_2 \otimes \boldsymbol{b}_1$ of the semi-infinite tensor product of B is called a reference path if it satisfies $\boldsymbol{b}_j \in B_{\min}$ and $\varphi(\boldsymbol{b}_{j+1}) = \varepsilon(\boldsymbol{b}_j)$ for any $j \geq 1$.*

Definition 3.4. *Fix a reference path $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_j \otimes \cdots \otimes \boldsymbol{b}_2 \otimes \boldsymbol{b}_1$. We define a set of paths $\mathcal{P}(\boldsymbol{p}, B)$ by*

$$\mathcal{P}(\boldsymbol{p}, B) = \{p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1 \mid b_j \in B, b_k = \boldsymbol{b}_k \text{ for } k \gg 1\}.$$

An element of $\mathcal{P}(\boldsymbol{p}, B)$ is called a *path*. For convenience we denote b_k by $p(k)$ and $\cdots \otimes b_{k+2} \otimes b_{k+1}$ by $p[k]$ for $p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1$.

Definition 3.5. *For a path $p \in \mathcal{P}(\boldsymbol{p}, B)$, set*

$$\begin{aligned} E(p) &= \sum_{j=1}^{\infty} j(H(p(j+1) \otimes p(j)) - H(\boldsymbol{p}(j+1) \otimes \boldsymbol{p}(j))), \\ W(p) &= \varphi(\boldsymbol{p}(1)) + \sum_{j=1}^{\infty} (\mathrm{wt}\, p(j) - \mathrm{wt}\, \boldsymbol{p}(j)) - E(p)\delta. \end{aligned}$$

$E(p)$ and $W(p)$ are called the energy and weight of p.

We distinguish $W(p) \in P$ from $wt\, p = \varphi(\boldsymbol{p}(1)) + \sum_{j=1}^{\infty}(wt\, p(j) - wt\, \boldsymbol{p}(j)) \in P_{cl}$.

REMARK 3.6. (i) *If B is perfect, the set of reference paths is bijective to $(P_{cl}^{+})_l$, where $l = \mathrm{lev}\, B$. For $\lambda \in (P_{cl}^{+})_l$ take a unique $\boldsymbol{b}_1 \in B_{\min}$ such that $\varphi(\boldsymbol{b}_1) = \lambda$. The condition $\varphi(\boldsymbol{b}_{j+1}) = \varepsilon(\boldsymbol{b}_j)$ fixes $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_j \otimes \cdots \otimes \boldsymbol{b}_1$ uniquely.*

(ii) *In [**KMN1**] $\boldsymbol{p}$ is called a ground state path, since $E(p) \geq E(\boldsymbol{p})$ for any $p \in \mathcal{P}(\boldsymbol{p}, B)$. But if B is not perfect, it is no longer true in general.*

The following theorem is essential for our consideration below.

THEOREM 3.7. *Assume* rank $\mathfrak{g} > 2$. *Then $\mathcal{P}(\boldsymbol{p}, B)$ is an object of $\mathcal{C}^h$.*

Proof. Assume $\tilde{e}_i p = \cdots \otimes \tilde{e}_i b_j \otimes \cdots \otimes b_1 \neq 0$. Note that $E(\tilde{e}_i p) = E(p) - \delta_{i0}$ and $wt\, \tilde{e}_i b_j = wt\, b_j + \alpha_i - \delta_{i0}\delta \in P_{cl}$. By Definition 3.5 it is immediate to see $\mathcal{P}(\boldsymbol{p}, B)$ is a P-weighted crystal. Thus one has to check the following:

(i) If for any $i, j \in I$ $(i \neq j)$, $\mathcal{P}(\boldsymbol{p}, B)$ regarded as $\{i,j\}$-crystal is a disjoint union of crystals of integrable highest weight modules over $U_q(\mathfrak{g}_{\{i,j\}})$.

(ii) For any $p \in \mathcal{P}(\boldsymbol{p}, B)$, there exist $l \geq 0, i_1, \cdots, i_l \in I$ such that $p' = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p \in \mathcal{P}(\boldsymbol{p}, B)$ is a highest weight element.

We prove (i) first. For $p \in \mathcal{P}(\boldsymbol{p}, B)$ take m, m' such that $p(k) = \boldsymbol{p}(k)$ for $k > m$ and $m' \gg m$. Note that if $\tilde{f}_{i_N} \cdots \tilde{f}_{i_1} p[m] = p[m'] \otimes b'_{m'} \otimes \cdots \otimes b'_{m+1}$, then $b'_k = \boldsymbol{p}(k)$ for $k > m + N$. From the assumption, $U_q(\mathfrak{g}_{\{i,j\}})$ is the quantized enveloping algebra associated to a finite-dimensional Lie algebra. Since B is regular, the connected component containing $p[m]$, as $\{i,j\}$-crystal, can be considered to be in $B(\varphi(p[m'])) \otimes B^{\otimes(m'-m)}$. Since $\varepsilon(p[m]) = 0$, we can regard $p[m]$ as highest weight element of some $\{i,j\}$-crystal B_0 which is isomorphic to the crystal of an integrable highest weight $U_q(\mathfrak{g}_{\{i,j\}})$-module. Hence p is contained in a component of the $\{i,j\}$-crystal $B_0 \otimes B^{\otimes m}$, which is a disjoint union of crystals of integrable highest weight $U_q(\mathfrak{g}_{\{i,j\}})$-modules.

To prove (ii) for $p = \cdots \otimes b_k \otimes \cdots \otimes b_1 \in \mathcal{P}(\boldsymbol{p}, B)$, we take the minimum integer m such that $p' = p[m]$ is a highest weight element. We prove by induction on m.

First let us show that there exist $l \geq 0, i_1, \cdots, i_l \in I$ such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l}(p' \otimes b_m)$ is a highest weight element. The proof is essentially the same as a part of that of Theorem 4.4.1 in [**KMN1**]. Nevertheless we repeat it for the sake of self-containedness. Suppose that there does not exist such $i_1, \cdots, i_l$. Then there exists an infinite sequence $\{i_\nu\}$ in I such that

$$\tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(p' \otimes b_m) \neq 0.$$

Since $\tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(p' \otimes b_m) = p' \otimes \tilde{e}_{i_k} \cdots \tilde{e}_{i_1} b_m$ and B is a finite set, there exists $b^{(1)} \in B$ and $j_1, \cdots, j_l$ such that

$$p' \otimes b^{(1)} = \tilde{e}_{j_l} \cdots \tilde{e}_{j_1}(p' \otimes b^{(1)}).$$

Hence setting $b^{(\nu+1)} = \tilde{e}_{j_\nu} b^{(\nu)}$, we have

$$\tilde{e}_{j_\nu}(p' \otimes b^{(\nu)}) = p' \otimes b^{(\nu+1)} \text{ and } b^{(l+1)} = b^{(1)}.$$

In view of (2.6) we have $\varphi_i(p') \geq \varphi_i(b_{m+1})$ for any i. Thus by (2.3) we have $\varepsilon_{j_\nu}(b^{(\nu)}) > \varphi_{j_\nu}(p') \geq \varphi_{j_\nu}(b')$ for some $b' \in B$. Hence we have

$$\tilde{e}_{j_\nu}(b' \otimes b^{(\nu)}) = b' \otimes b^{(\nu+1)}.$$

Therefore, from (3.2), we have

$$H(b' \otimes b^{(\nu+1)}) = H(b' \otimes b^{(\nu)}) - \delta_{i_\nu 0}.$$

Hence $H(b' \otimes b^{(l+1)}) = H(b' \otimes b^{(1)}) - \sharp\{\nu \mid j_\nu = 0\}$, which implies there is no ν such that $j_\nu = 0$. On the other hand, $\sum_\nu \alpha_{j_\nu} = 0 \bmod \mathbf{Z}\delta$ and hence $\sum_\nu \alpha_{j_\nu}$ is a positive multiple of δ, which contradicts $0 \notin \{j_1, \cdots, j_l\}$.

Now set $p'' = p' \otimes b_m (= p[m-1]), b'' = b_{m-1} \otimes \cdots \otimes b_1$. Notice that for any $i \in I$ satisfying $\tilde{e}_i p'' \neq 0$, there exists $k \geq 1$ such that

$$\tilde{e}_i^k (p'' \otimes b'') = \tilde{e}_i p'' \otimes \tilde{e}_i^{k-1} b''.$$

Therefore there exist $l \geq 0, (i_1, k_1), \cdots, (i_l, k_l) \in I \times \mathbf{Z}_{>0}$ such that

$$\tilde{e}_{i_1}^{k_1} \cdots \tilde{e}_{i_l}^{k_l} p = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p'' \otimes \tilde{e}_{i_1}^{k_1 - 1} \cdots \tilde{e}_{i_l}^{k_l - 1} b''$$

and $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p''$ is a highest weight element. Now we can use the induction assumption and complete the proof. ▮

Remark 3.8. *As seen in the proof, the theorem does not require the condition $\boldsymbol{b}_j \in B_{\min}$ for the reference path $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_j \otimes \cdots \otimes \boldsymbol{b}_1$.*

The following proposition describes the set of highest weight elements in $\mathcal{P}(\boldsymbol{p}, B)$.

Proposition 3.9.

$$\mathcal{P}(\boldsymbol{p}, B)_0 = \{p \in \mathcal{P}(\boldsymbol{p}, B) \mid p(j) \in B_{\min}, \varphi(p(j+1)) = \varepsilon(p(j)) \text{ for } \forall j\}.$$

Proof. Assume $p = \cdots \otimes b_j \otimes \cdots \otimes b_1$ is a highest weight element. We prove the following by induction on m in decreasing order.

(i) $b_m \in B_{\min}, \varphi(b_{m+1}) = \varepsilon(b_m)$
(ii) $\varphi(p[m-1]) = \varphi(b_m)$

These conditions are satisfied for sufficiently large m. From (ii) for $m+1$ we have $\varphi(p[m]) = \varphi(b_{m+1})$. From Lemma 2.11 we see that $p[m]$ is a highest weight element and $\varepsilon(b_m) \leq wt\, p[m] = \varphi(p[m]) = \varphi(b_{m+1})$. Combining this with (i) for $m+1$, we can conclude (i) for m. For (ii) use (2.6). ▮

As seen in the proof, we obtain

Corollary 3.10. *If $p \in \mathcal{P}(\boldsymbol{p}, B)_0$, then $wt\, p[j] = \varphi(p(j+1))$.*

3.3. Restricted paths. When B is perfect the set of *restricted* paths was defined in [**DJO**] and shown to be bijective to $(B(\lambda) \otimes B(\mu))_0$ for some $\lambda, \mu \in P_{cl}^+$. Here we shall consider restricted paths for any finite crystal B.

For $\lambda \in P_{cl}^+$ and $p \in \mathcal{P}(\boldsymbol{p}, B)$, we introduce a sequence of weights $\{\lambda_j(p)\}_{j \geq 0}$ by

$$\lambda_j(p) = \lambda + \varphi(p(j+1)) \text{ for } j \gg 1,$$
$$\lambda_{j-1}(p) = \lambda_j(p) + wt\, p(j).$$

Notice that this definition is well-defined by virtue of the property of the reference path. In fact, $\lambda_j(p) = \lambda + wt\, p[j]$.

Definition 3.11. *For $\lambda \in P_{cl}^+$ we define a subset $\mathcal{P}^{(\lambda)}(\boldsymbol{p}, B)$ of $\mathcal{P}(\boldsymbol{p}, B)$ by*

$$\mathcal{P}^{(\lambda)}(\boldsymbol{p}, B) = \{p \in \mathcal{P}(\boldsymbol{p}, B) \mid \tilde{e}_i^{\langle h_i, \lambda_j(p) \rangle + 1} p(j) = 0 \text{ for } \forall i, j\}.$$

An element of $\mathcal{P}^{(\lambda)}(\boldsymbol{p}, B)$ is called a *restricted* path.

Proposition 3.12. *For $\lambda \in P_{cl}^+$ we have*

$$\mathcal{P}(\boldsymbol{p}, B)^{\leq\lambda} = \mathcal{P}^{(\lambda)}(\boldsymbol{p}, B).$$

Proof. Assume $p = \cdots \otimes b_j \otimes \cdots \otimes b_1 \in \mathcal{P}(\boldsymbol{p}, B)^{\leq\lambda}$, which is equivalent to saying $u_\lambda \otimes p$ is a highest weight element. So is $u_\lambda \otimes p[j] \otimes b_j$ by Lemma 2.11. Using this lemma again we get $\varepsilon(b_j) \leq wt\,(u_\lambda \otimes p[j]) = \lambda_j(p)$.

To show the inverse inclusion, assume $p = \cdots \otimes b_j \otimes \cdots \otimes b_1 \in \mathcal{P}^{(\lambda)}(\boldsymbol{p}, B)$. We prove $\varepsilon(p[j]) \leq \lambda$ by induction on j in decreasing order. We know $\varepsilon(p[j]) = 0$ for sufficiently large j. Supposing $\varepsilon(p[j]) \leq \lambda$ we immediately obtain $\varepsilon(p[j] \otimes b_j) \leq \lambda$ from (2.5) and the condition $\varepsilon(b_j) \leq \lambda_j(p)$. ∎

As seen in the proof we have $\lambda_j(p) \in P_{cl}^+$ and its level is $\langle c, \lambda\rangle + lev\, B$.

Combining the results in section 2.4, Theorem 3.7 and Proposition 3.12, we obtain

Theorem 3.13. *Let $\mathcal{P}(\boldsymbol{p}, B)$ and $\mathcal{P}(\boldsymbol{p}^\dagger, B^\dagger)$ be two sets of paths. If for certain $\lambda \in P_{cl}^+$, there exists a bijection*

$$\begin{array}{ccc} \mathcal{P}(\boldsymbol{p}, B)_0 & \longrightarrow & \mathcal{P}^{(\lambda)}(\boldsymbol{p}^\dagger, B^\dagger) \\ p & \mapsto & p^\dagger \end{array} \tag{3.3}$$

such that $W(p) = \lambda + W(p^\dagger)$, then we have an isomorphism of P-weighted crystals

$$\mathcal{P}(\boldsymbol{p}, B) \simeq B(\lambda) \otimes \mathcal{P}(\boldsymbol{p}^\dagger, B^\dagger).$$

They are isomorphic to a direct sum of crystals of integrable highest weight $U_q(\mathfrak{g})$-modules, and their highest weight elements are parametrized by (3.3).

4. Examples

We shall give two examples to which we can apply Theorem 3.13 efficiently.

4.1. Example 1. We present a useful proposition first. Similar to $O^{\leq\lambda}$ we define $B^{\leq\lambda}$ for a finite crystal B and $\lambda \in P_{cl}^+$ by

$$B^{\leq\lambda} = \{b \in B \mid \tilde{e}_i^{\langle h_i, \lambda\rangle+1} b = 0 \text{ for any } i\}.$$

Note that if $lev\, B = l$, then $B_{\min} = \bigsqcup_{\lambda \in (P_{cl}^+)_l} B^{\leq\lambda}$.

Proposition 4.1. *Let B and $B^\dagger$ be finite crystals such that $\mathrm{lev}\, B \geq \mathrm{lev}\, B^\dagger$, and $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_j \otimes \cdots \otimes \boldsymbol{b}_1$ be a reference path for B. Suppose there exists a map $t : B_{\min} \to B^\dagger$ satisfying the following conditions:*

1. *For any $\mu \in (P_{cl}^+)_l$ $(l = \mathrm{lev}\, B)$, $t|_{B^{\leq\mu}}$ is a bijection onto $(B^\dagger)^{\leq\mu}$.*
2. *$wt\, t(b) = wt\, b$ for any $b \in B_{\min}$.*
3. *$H_{B^\dagger B^\dagger}(t(b_1) \otimes t(b_2)) = H_{BB}(b_1 \otimes b_2)$ up to global additive constant for any $(b_1, b_2) \in B_{\min}^2$ such that $\varphi(b_1) = \varepsilon(b_2)$.*
4. *$\boldsymbol{p}^\dagger = \cdots \otimes t(\boldsymbol{b}_j) \otimes \cdots \otimes t(\boldsymbol{b}_1)$ is a reference path for $B^\dagger$.*

Then setting $\lambda = \varphi(\boldsymbol{b}_1) - \varphi(t(\boldsymbol{b}_1))$, we have

$$\mathcal{P}(\boldsymbol{p}, B) \simeq B(\lambda) \otimes \mathcal{P}(\boldsymbol{p}^\dagger, B^\dagger).$$

Proof. Consider the following map.

$$\begin{array}{ccc} \mathcal{P}(\boldsymbol{p}, B)_0 & \longrightarrow & \mathcal{P}(\boldsymbol{p}^\dagger, B^\dagger) \\ p = \cdots \otimes b_j \otimes \cdots \otimes b_1 & \mapsto & p^\dagger = \cdots \otimes t(b_j) \otimes \cdots \otimes t(b_1) \end{array}$$

From Theorem 3.13 it suffices to show that this map is a bijection onto $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^\dagger, B^\dagger)$ such that $W(p) = \lambda + W(p^\dagger)$. Preservation of weight is immediate. To show the bijectivity one has to notice that $wt\, p^\dagger[j] - wt\, p[j]$ does not depend on j. Thus one has $wt\, p^\dagger[j] - wt\, p[j] = wt\, p^\dagger - wt\, p = -\lambda$, and hence

$$\lambda_j(p^\dagger) = \lambda + wt\, p^\dagger[j] = wt\, p[j] = \varphi(b_{j+1}) = \varepsilon(b_j).$$

Note that $p \in \mathcal{P}(\boldsymbol{p}, B)_0$ (cf. Proposition 3.9 & Corollary 3.10). In view of (1) this equality concludes the bijectivity. ∎

We now consider the $C_n^{(1)}$ case. For an odd positive integer l, consider a finite crystal $B^{1,l}$ given by

$$B^{1,l} = \left\{ (x_1, \dots, x_n, \overline{x}_n, \dots, \overline{x}_1) \;\middle|\; \begin{array}{l} x_i, \overline{x}_i \in \mathbf{Z}_{\geq 0}\; \forall\, i = 1, \cdots, n \\ \sum_{i=1}^n (x_i + \overline{x}_i) \in \{l, l-2, \dots, 1\} \end{array} \right\}.$$

The crystal structure of $B^{1,l}$ is given by

$$\begin{aligned}
\tilde{e}_0 b &= \begin{cases} (x_1 - 2, x_2, \dots, \overline{x}_2, \overline{x}_1) & \text{if } x_1 \geq \overline{x}_1 + 2, \\ (x_1 - 1, x_2, \dots, \overline{x}_2, \overline{x}_1 + 1) & \text{if } x_1 = \overline{x}_1 + 1, \\ (x_1, x_2, \dots, \overline{x}_2, \overline{x}_1 + 2) & \text{if } x_1 \leq \overline{x}_1, \end{cases} \\
\tilde{e}_i b &= \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \overline{x}_1) & \text{if } x_{i+1} > \overline{x}_{i+1}, \\ (x_1, \dots, \overline{x}_{i+1} + 1, \overline{x}_i - 1, \dots, \overline{x}_1) & \text{if } x_{i+1} \leq \overline{x}_{i+1}, \end{cases} \\
\tilde{e}_n b &= (x_1, \dots, x_n + 1, \overline{x}_n - 1, \dots, \overline{x}_1), \\
\tilde{f}_0 b &= \begin{cases} (x_1 + 2, x_2, \dots, \overline{x}_2, \overline{x}_1) & \text{if } x_1 \geq \overline{x}_1, \\ (x_1 + 1, x_2, \dots, \overline{x}_2, \overline{x}_1 - 1) & \text{if } x_1 = \overline{x}_1 - 1 \\ (x_1, x_2, \dots, \overline{x}_2, \overline{x}_1 - 2) & \text{if } x_1 \leq \overline{x}_1 - 2, \end{cases} \\
\tilde{f}_i b &= \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \overline{x}_1) & \text{if } x_{i+1} \geq \overline{x}_{i+1}, \\ (x_1, \dots, \overline{x}_{i+1} - 1, \overline{x}_i + 1, \dots, \overline{x}_1) & \text{if } x_{i+1} < \overline{x}_{i+1}, \end{cases} \\
\tilde{f}_n b &= (x_1, \dots, x_n - 1, \overline{x}_n + 1, \dots, \overline{x}_1),
\end{aligned}$$

where $b = (x_1, \dots, x_n, \overline{x}_n, \dots, \overline{x}_1)$ and $i = 1, \dots, n-1$. If some component becomes negative upon application, it should be understood as 0. The values of ε_i, φ_i read

$$\begin{array}{ll}
\varepsilon_0(b) = \frac{l - s(b)}{2} + (x_1 - \overline{x}_1)_+, & \varphi_0(b) = \frac{l - s(b)}{2} + (\overline{x}_1 - x_1)_+, \\
\varepsilon_i(b) = \overline{x}_i + (x_{i+1} - \overline{x}_{i+1})_+, & \varphi_i(b) = x_i + (\overline{x}_{i+1} - x_{i+1})_+, \\
\varepsilon_n(b) = \overline{x}_n, & \varphi_n(b) = x_n.
\end{array}$$

Here $s(b) = \sum_{i=1}^n (x_i + \overline{x}_i), (x)_+ = \max(x, 0)$ and $i = 1, \cdots, n-1$. $B^{1,l}$ is a level $\frac{l+1}{2}$ non-perfect crystal. Now for a fixed l set $B = B^{1,l}$. The minimal elements of B are grouped as $B_{\min} = \bigsqcup_{\mu \in (P_{cl}^+)_{\frac{l+1}{2}}} B^{\leq \mu}$, where for $\mu = \mu_0 \Lambda_0 + \cdots + \mu_n \Lambda_n$. The set $B^{\leq \mu}$ is given by

$$\begin{aligned}
B^{\leq \mu} &= \{ b_k^\mu \mid \mu_{k-1} > 0, 1 \leq k \leq n \} \cup \{ b_{\overline{k}}^\mu \mid \mu_k > 0, 1 \leq k \leq n \}, \\
b_k^\mu &= (\mu_1, \dots, \mu_{k-1} - 1, \mu_k + 1, \dots, \mu_n, \mu_n, \dots, \mu_{k-1} - 1, \dots, \mu_1), \\
b_{\overline{k}}^\mu &= (\mu_1, \dots, \mu_k - 1, \dots, \mu_n, \mu_n, \dots, \mu_1).
\end{aligned}$$

Next consider $B^\dagger = B^{1,1}$ by taking l to be 1. Setting

$$b_k^\dagger = (x_i = \delta_{ik}, \overline{x}_i = 0), \quad b_{\overline{k}}^\dagger = (x_i = 0, \overline{x}_i = \delta_{ik})$$

for $1 \le k \le n$, one has

$$(B^\dagger)^{\le\mu} = \{b^\dagger_k \mid \mu_{k-1} > 0, 1 \le k \le n\} \cup \{b^\dagger_{\overline{k}} \mid \mu_k > 0, 1 \le k \le n\}$$

for μ as above. Define the map $t : B_{\min} \to B^\dagger$ by

$$t|_{B^{\le\mu}} : b^\mu_k \mapsto b^\dagger_k \qquad \text{for } k \in \{1, \dots, n, \overline{n}, \dots, \overline{1}\}.$$

We are to show that this t satisfies the conditions (1) – (4) in Proposition 4.1. For our purpose fix a dominant integral weight $\lambda \in (P^+_{cl})_{\frac{l-1}{2}}$ and define $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_j \otimes \cdots \otimes \boldsymbol{b}_1$ by

$$\boldsymbol{b}_j = \begin{cases} b^{\lambda+\Lambda_i}_{\overline{i}} & \text{if } j \equiv i \,(\mathrm{mod}\; 2n) \text{ for some } i \,(1 \le i \le n), \\ b^{\lambda+\Lambda_{i-1}}_{i} & \text{if } j \equiv 1-i \,(\mathrm{mod}\; 2n) \text{ for some } i \,(1 \le i \le n). \end{cases}$$

Note that $\varepsilon(b^{\lambda+\Lambda_i}_{\overline{i}}) = \varphi(b^{\lambda+\Lambda_{i-1}}_{i}) = \lambda + \Lambda_i, \varepsilon(b^{\lambda+\Lambda_{i-1}}_{i}) = \varphi(b^{\lambda+\Lambda_i}_{\overline{i}}) = \lambda + \Lambda_{i-1}$. $\boldsymbol{p}$ becomes a reference path. Let us check (1) – (4) in Proposition 4.1. (1),(2) and (4) are straightforward. To check (3) one can use the formula for H_{BB} in [**KKM**] section 5.7. (In [**KKM**] our non-perfect case is not considered. However, the formula itself is valid. Since the formula in [**KKM**] contains some misprints, we rewrite it below.)

$$H_{B^{1,l}B^{1,l}}(b \otimes b') = \max_{1 \le j \le n} (\theta_j(b \otimes b'), \theta'_j(b \otimes b'), \eta_j(b \otimes b'), \eta'_j(b \otimes b')),$$

$$\begin{aligned}
\theta_j(b \otimes b') &= \sum_{k=1}^{j-1}(\overline{x}_k - \overline{x}'_k) + \frac{1}{2}(s(b') - s(b)), \\
\theta'_j(b \otimes b') &= \sum_{k=1}^{j-1}(x'_k - x_k) + \frac{1}{2}(s(b) - s(b')), \\
\eta_j(b \otimes b') &= \sum_{k=1}^{j-1}(\overline{x}_k - \overline{x}'_k) + (\overline{x}_j - x_j) + \frac{1}{2}(s(b') - s(b)), \\
\eta'_j(b \otimes b') &= \sum_{k=1}^{j-1}(x'_k - x_k) + (x'_j - \overline{x}'_j) + \frac{1}{2}(s(b) - s(b')),
\end{aligned}$$

where $b = (x_1, \dots, x_n, \overline{x}_n, \dots, \overline{x}_1), b' = (x'_1, \dots, x'_n, \overline{x}'_n, \dots, \overline{x}'_1)$.

Therefore, the isomorphism in Proposition 4.1 holds with notations above.

4.2. Example 2. We consider the $A^{(1)}_{n-1}$ case. Let $B^{1,l}$ be the crystal base of the symmetric tensor representation of $U'_q(A^{(1)}_{n-1})$ of degree l. As a set it reads

$$B^{1,l} = \{(a_0, a_1, \cdots, a_{n-1}) \mid a_i \in \mathbf{Z}_{\ge 0}, \sum_{i=0}^{n-1} a_i = l\}.$$

For convenience we extend the definition of a_i to $i \in \mathbf{Z}$ by setting $a_{i+n} = a_i$ and use a simpler notation (a_i) for $(a_0, a_1, \cdots, a_{n-1})$. For instance, (a_{i-1}) means $(a_{n-1}, a_0, \cdots, a_{n-2})$. The actions of $\tilde{e}_r, \tilde{f}_r$ $(r = 0, \cdots, n-1)$ are given by

$$\tilde{e}_r(a_i) = (a_i - \delta^{(n)}_{i,r} + \delta^{(n)}_{i,r-1}), \quad \tilde{f}_r(a_i) = (a_i + \delta^{(n)}_{i,r} - \delta^{(n)}_{i,r-1}).$$

Here $\delta_{ij}^{(n)} = 1$ ($i \equiv j \bmod n$), $= 0$ (otherwise). If some component becomes negative upon application, it should be understood as 0. The values of ε, φ read as follows.

$$\varepsilon((a_i)) = \sum_{i=0}^{n-1} a_i \Lambda_i, \quad \varphi((a_i)) = \sum_{i=0}^{n-1} a_{i-1} \Lambda_i.$$

Thus $lev\, B^{1,l} = l$ and all elements are minimal. We introduce a $\mathbf{Z}$-linear automorphism σ on P_{cl} by $\sigma \Lambda_i = \Lambda_{i-1}$ ($\Lambda_{-1} = \Lambda_{n-1}$).

Now consider the finite crystal $B = B^{1,l} \otimes B^{1,m}$ ($l \geq m$) and set $B^\dagger = B^{1,m}$. From Lemma 2.7 (1) the level of B is l. Fix two dominant integral weights $\lambda = \sum_{i=0}^{n-1} \lambda_i \Lambda_i \in (P_{cl}^+)_{l-m}, \mu = \sum_{i=0}^{n-1} \mu_i \Lambda_i \in (P_{cl}^+)_m$. From (λ, μ) we define a path

$$\boldsymbol{p}^{(\lambda,\mu)}(j) = (\lambda_{i+j} + \mu_{i+2j}) \otimes (\mu_{i+2j-1}) \in B.$$

From Lemma 2.7 (2) we see $\boldsymbol{p}^{(\lambda,\mu)}(j) \in B_{\min}$ and by (2.5),(2.6) we obtain $\varepsilon(\boldsymbol{p}^{(\lambda,\mu)}(j)) = \sigma^j \lambda + \sigma^{2j} \mu = \varphi(\boldsymbol{p}^{(\lambda,\mu)}(j+1))$. Therefore $\boldsymbol{p}^{(\lambda,\mu)}$ is a reference path.

We would like to show

$$\mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B) \simeq B(\lambda) \otimes \mathcal{P}(\boldsymbol{p}^{(\mu)}, B^\dagger) \quad \text{as } P\text{-weighted crystals} \tag{4.1}$$

with $\boldsymbol{p}^{(\mu)}(j) = (\mu_{i+j})$. To do this, consider the following map

$$\begin{array}{ccc} \mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B)_0 & \longrightarrow & \mathcal{P}(\boldsymbol{p}^{(\mu)}, B^\dagger) \\ p & \mapsto & p^\dagger \end{array} \tag{4.2}$$

given by $p^\dagger(j) = (b_{i-j+1}^{(j)})$ for $p(j) = (a_i^{(j)}) \otimes (b_i^{(j)})$. Note that $\boldsymbol{p}^{(\lambda,\mu)}$ is sent to $\boldsymbol{p}^{(\mu)}$ under this map. By Theorem 3.13 it suffices to check the following items:

(i) The map (4.2) is a bijection onto $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^{(\mu)}, B^\dagger)$.
(ii) $wt\, p - wt\, p^\dagger = \lambda$.
(iii) $E(p) = E(p^\dagger)$.

Since $p \in \mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B)_0$, one obtains (cf. Lemma 2.7 (2), Proposition 3.9)

$$\varphi_i((a_i^{(j)})) = a_{i-1}^{(j)} \geq b_i^{(j)} = \varepsilon_i((b_i^{(j)})) \tag{4.3}$$

$$\varphi_i(p(j)) = a_{i-1}^{(j)} + b_{i-1}^{(j)} - b_i^{(j)} = a_i^{(j-1)} = \varepsilon_i(p(j-1)) \tag{4.4}$$

for any i, j. Taking sufficiently large J and using (4.4), one has

$$\begin{aligned} wt\, p^\dagger[j] &= \sum_i b_{i-J+1}^{(J)} \Lambda_i + \sum_{k=j+1}^{J} \sum_i (b_{i-k}^{(k)} - b_{i-k+1}^{(k)}) \Lambda_i \\ &= \sum_i (b_{i-J+1}^{(J)} - a_{i-J}^{(J)} + a_{i-j}^{(j)}) \Lambda_i \\ &= \sum_i a_{i-j}^{(j)} \Lambda_i - \lambda. \end{aligned}$$

Thus the condition $\varepsilon(p^\dagger(j)) \leq \lambda_j(p^\dagger)$ is equivalent to saying $b_{i-j+1}^{(j)} \leq a_{i-j}^{(j)}$ for any i, which is guaranteed by (4.3). This proves (i). For (ii) one only has to notice that $wt\, p[j] = \varphi(p(j+1)) = \sum_i a_i^{(j)} \Lambda_i$.

In order to prove (iii), we set

$$E_L^{diff} = \sum_{j=1}^{L} j\big\{H_{BB}(((a_i^{(j+1)}) \otimes (b_i^{(j+1)})) \otimes ((a_i^{(j)}) \otimes (b_i^{(j)}))) - H_{B^\dagger B^\dagger}((b_{i-(j+1)+1}^{(j+1)}) \otimes (b_{i-j+1}^{(j)}))\big\}.$$

We can assume $(a_i^{(j)}) \otimes (b_i^{(j)}) \in B_{\min}$ for $1 \le j \le L+1$. Under such assumption the isomorphism $B^{1,l} \otimes B^{1,m} \simeq B^{1,m} \otimes B^{1,l}$ sends $(a_i) \otimes (b_i)$ to $(b_{i+1}) \otimes (a_i - b_{i+1} + b_i)$ [**NY**]. Thus, from Proposition 3.1 we have

$$H_{BB}(((a_i) \otimes (b_i)) \otimes ((a'_i) \otimes (b'_i))) = b_0 + a'_0 + b'_0 + H_{B^\dagger B^\dagger}((b_i) \otimes (b'_{i+1})).$$

Let us recall the following formula for $H_{B^{1,m}B^{1,m}}$ (cf. [**KKM**] section 5.1).

$$H_{B^{1,m}B^{1,m}}((b_i) \otimes (b'_i)) = \max_{0 \le j \le n-1} \big(\sum_{k=0}^{j-1}(b'_k - b_k) + b'_j\big)$$

From this one gets

$$H_{B^\dagger B^\dagger}((b_i^{(j+1)}) \otimes (b_{i+1}^{(j)})) - H_{B^\dagger B^\dagger}((b_{i-j}^{(j+1)}) \otimes (b_{i-j+1}^{(j)})) = \sum_{k=1}^{j}(b_{k-j-1}^{(j+1)} - b_{k-j}^{(j)}).$$

Using above facts and (4.4) one obtains

$$E_L^{diff} = \sum_{j=1}^{L}\sum_{k=0}^{j-1} a_{-k}^{(L)} + L\sum_{k=0}^{L} b_{-k}^{(L+1)}.$$

This completes (iii). We have finished proving (4.1). It is also known [**KMN2**] that $\mathcal{P}(\boldsymbol{p}^{(\mu)}, B^{1,m}) \simeq B(\mu)$. Therefore we have

$$\mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B^{1,l} \otimes B^{1,m}) \simeq B(\lambda) \otimes B(\mu) \quad \text{as } P\text{-weighted crystals.}$$

The multi-component version is straightforward. Consider the finite crystal $B^{1,l_1} \otimes \cdots \otimes B^{1,l_s}$ $(l_1 \ge \cdots \ge l_s \ge l_{s+1} = 0)$. For $\lambda^{(i)} \in (P_{cl}^+)_{l_i - l_{i+1}}$ $(1 \le i \le s)$ we define a reference path $\boldsymbol{p}^{(\lambda_1,\cdots,\lambda_s)}$ by

$$\text{the } k\text{-th tensor component of } \boldsymbol{p}^{(\lambda_1,\cdots,\lambda_s)}(j) = (\lambda_{i+kj-k+1}^{(k)} + \lambda_{i+(k+1)j-k+1}^{(k+1)} + \cdots + \lambda_{i+sj-k+1}^{(s)}).$$

Then we have

$$\mathcal{P}(\boldsymbol{p}^{(\lambda_1,\cdots,\lambda_s)}, B^{1,l_1} \otimes \cdots \otimes B^{1,l_s}) \simeq B(\lambda_1) \otimes \cdots \otimes B(\lambda_s).$$

The proof will be given elsewhere.

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